

**CONCURSUL PLURIDISCIPLINAR PROSOFT@NT**

martie 2017

**MATH-Individual Contest
Solutions-CLASA a IX a****Problem 1**If $a, b, c \in \mathbb{R}_+^*$ si $a + b + c = 1$ prove:

a). $(a + bc)(b + ac)(c + ab) = (a + b)^2(b + c)^2(c + a)^2$

b). $\left(\frac{a}{bc} + 1\right)\left(\frac{b}{ac} + 1\right)\left(\frac{c}{ab} + 1\right) \geq 64$

Solutions problem 1a). $a + b + c = 1$. We multiply the relation with b and we add c, obtaining

$$ab + c = (b + c)(1 - b) \Rightarrow ab + c = (b + c)(a + c) \text{ and analogous} \quad (10p)$$

Obtaining the equality

(3p)

b). Using the previous point, the equality is equivalent with

$$\left[\frac{(a + b)(b + c)(c + a)}{abc} \right]^2 \geq 64 \quad (7p)$$

$$a + b \geq 2\sqrt{ab} \text{ and the analogous} \quad (3p)$$

Replacing and Obtaining the equality (2p)

Problem 2Be M a point inside $\triangle ABC$ non-isosceles of sides a,b,c with G the barycenter (center of gravity) of the triangle and I the center of the incircle.

a) Demonstrate that $\frac{AM}{MD} = \frac{A_{\triangle ABM} + A_{\triangle ACM}}{A_{\triangle BCM}}$ where $\{D\} = AM \cap BC$.

b) Demonstrate that M,G,I are collinear if and only if

$$(b - c)A_{\triangle BCM} + (c - a)A_{\triangle CAM} + (a - b)A_{\triangle ABM} = 0$$

Solutions problem 2

a). From $\frac{AM}{MD} = \frac{A_{\triangle ABM}}{A_{\triangle BMD}} = k$ si $\frac{AM}{MD} = \frac{A_{\triangle AMC}}{A_{\triangle CMD}} = k \quad (7p)$

and $A_{\triangle BCM} = A_{\triangle BMD} + A_{\triangle CMD}$ hence the result appears (3p)

b). If $BT \perp AD, CQ \perp AD$ ($T, Q \in AD$) $\Rightarrow \frac{BD}{DC} = \frac{BT}{CQ} = \frac{A_{\triangle ABM}}{A_{\triangle ACM}} = p \quad (7p)$

$$\vec{OM} = \frac{1}{k+1} \vec{OA} + \frac{k}{k+1} \vec{OD} \text{ si } \vec{OD} = \frac{1}{p+1} \vec{OB} + \frac{p}{p+1} \vec{OC} \text{ we deduce that}$$

$$\vec{OM} = \frac{A_{\triangle BCM}}{A_{\triangle ABC}} \vec{OA} + \frac{A_{\triangle ACM}}{A_{\triangle ABC}} \vec{OB} + \frac{A_{\triangle ABM}}{A_{\triangle ABC}} \vec{OC} \quad (3p)$$

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Cum $\vec{OI} = \frac{a}{a+b+c} \vec{OA} + \frac{b}{a+b+c} \vec{OB} + \frac{c}{a+b+c} \vec{OC}$ si $\vec{GA} + \vec{GB} + \vec{GC} = \vec{0}$ hence it results that

$$\vec{GM} = \frac{A_{\Delta BCM} - A_{\Delta ABM}}{A_{\Delta ABC}} \vec{GA} + \frac{A_{\Delta ACM} - A_{\Delta ABM}}{A_{\Delta ABC}} \vec{GB}$$

$$\vec{GI} = \frac{a-c}{a+b+c} \vec{GA} + \frac{b-c}{a+b+c} \vec{GB} \quad (3p)$$

$$M, G, I \text{ collinear} \Leftrightarrow \vec{GM} \text{ si } \vec{GI} \text{ coliniari} \Leftrightarrow \frac{A_{\Delta BCM} - A_{\Delta ABM}}{a-c} = \frac{A_{\Delta ACM} - A_{\Delta ABM}}{b-c}$$

Hence the result appears (2p)

Problem 3

Be $n \in \mathbb{N}^*$ determined.

a) Determine the rest of the division of 5^{2^n} to 2^{n+2} .

b) Show that the array (order) $a_n = \left(\left\{ \frac{5^{2^n}}{2^{n+2}} \right\} \right)^{-1}$, where $\{a\}$ is the fractionary part of the

real number a , is a geometric progression and that every other arithmetic progression $(b_n)_{n \geq 1}$ with natural terms that has two common terms with $(a_n)_{n \geq 1}$ has an infinity of common terms with the progression $(a_n)_{n \geq 1}$.

Solutions problem 3

a). $5^{2^n} = 5^{2^n} - 1 + 1 = (5-1)(5^{2^n-1} + 5^{2^n-2} + \dots + 5 + 1) + 1$ 5p

$5-1:4$ si $5^{2^k} + 1:2$ Hence the result appears 5p

b). $\left\{ \frac{5^{2^n}}{2^{n+2}} \right\} = \left\{ \frac{2^{n+2}k+1}{2^{n+2}} \right\} = \frac{1}{2^{n+2}} \Rightarrow \left\{ \frac{5^{2^n}}{2^{n+2}} \right\}^{-1} = 2^{n+2}$ so the geometric progression 3p

$b_s = a_p \Rightarrow b_s = 2^{n+p}$ (the _first _two _common _terms)

$b_s + mr = 2^{n+t}$ (the _other _two _common _terms) 2p

$mr = 2^{n+p}(2^{t-p} - 1) \Rightarrow r | 2^{n+p}(2^{t-p} - 1) \Rightarrow r | 2^{n+p}(2^{k(t-p)} - 1)$ 5p

$2^{n+p+kt-kp} - 2^{n+p} = ru \Rightarrow 2^{n+p} + ru = 2^{n+p+k(t-p)} \Rightarrow$

$b_s + ru = a_p 2^{k(t-p)} \Rightarrow$ other _common _terms 5p

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Problem 4

Consider the array $(x_n)_{n \geq 0}$ given by $x_n = 2^n - n$, $n \in \mathbb{N}$. Determine all the natural numbers p for which $s_p = x_0 + x_1 + x_2 + \dots + x_p$ is a degree with a natural exponent of 2.

Solutions problem 4

$$S_p = 2^{p+1} \cdot \frac{p(p+1)}{2} - 1 \quad 5p$$

It is shown that $2^p < s_p < 2^{p+1}$, for $p \geq 3$. s_p is not a degree with a natural exponent of 2. $3p$

It is proven through mathematical induction that $\frac{p(p+1)}{2} + 1 < 2^p$, for $p \geq 3$ $8p$

It is proven that $2^p < s_p$ $4p$

It is noticed that $s_0 = 2^0$, $s_1 = 2^1$, $s_2 = 2^2$ $3p$

Hence, $p \in \{0, 1, 2\}$ $2p$