CONCURSUL PLURIDISCIPLINAR PROSOFT@NT

martie 2017

MATH-Individual Contest
Solutions-CLASA a IX a

Problem 1

If $a,b,c \in R_{+}^{*}$ si $a+b+c=1$ prove:

a). $(a+bc)(b+ac)(c+ab)=(a+b)^2(b+c)^2(c+a)^2$

b). $\left(\frac{a}{bc}+1\right)\left(\frac{b}{ac}+1\right)\left(\frac{c}{ab}+1\right)\geq 64$

Solutions problem 1

a). $a+b+c=1$. We multiply the relation with b and we add c, obtaining

$ab+c=(b+c)(1-b)\Rightarrow ab+c=(b+c)(a+c)$ and analogous (10p)

Obtaining the equality (3p)

b).Using the previous point, the equality is equivalent with

$\left[\frac{(a+b)(b+c)(c+a)}{abc}\right]^2\geq 64$ (7p)

$a+b\geq 2\sqrt{ab}$ and the analogous (3p)

Replacing and Obtaining the equality (2p)

Problem 2

Be M a point inside ΔABC non-isosceles of sides a,b,c with G the barycenter (center of gravity) of the triangle and I the center of the incircle.

a)Demonstrate that $\frac{AM}{MD}=\frac{A_{\Delta ABM}+A_{\Delta ACM}}{A_{\Delta BCM}}$ where $\{D\}=AM\cap BC$.

b)Demonstrate that M,G,I are collinear if and only if

$(b-c)A_{\Delta BCM}+(c-a)A_{\Delta CAM}+(a-b)A_{\Delta ABM}=0$

Solutions problem 2

a).From $\frac{AM}{MD}=\frac{A_{\Delta ABM}}{A_{\Delta BMD}}=k$ si $\frac{AM}{MD}=\frac{A_{\Delta AMC}}{A_{\Delta CMD}}=k$ (7p)

and $A_{\Delta BCM}=A_{\Delta BMD}+A_{\Delta CMD}$ hence the result appears (3p)

b).If $BT\perp AD, CQ\perp AD$ $(T,Q\in AD)\Rightarrow\frac{BD}{DC}=\frac{BT}{CQ}=\frac{A_{\Delta ABM}}{A_{\Delta ACM}}=p$ (7p)

$\vec{OM}=\frac{1}{k+1}\vec{OA}+\frac{k}{k+1}\vec{OD}$ si $\vec{OD}=\frac{1}{p+1}\vec{OB}+\frac{p}{p+1}\vec{OC}$ we deduce that

$\vec{OM}=\frac{A_{\Delta BCM}}{A_{\Delta ABC}}\vec{OA}+\frac{A_{\Delta ACM}}{A_{\Delta ABC}}\vec{OB}+\frac{A_{\Delta ABM}}{A_{\Delta ABC}}\vec{OC}$ (3p)

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Cum $\vec{OI} = \frac{a}{a+b+c}\vec{OA} + \frac{b}{a+b+c}\vec{OB} + \frac{c}{a+b+c}\vec{OC}$ si $\vec{GA}+\vec{GB}+\vec{GC}=\vec{0}$ hence it results that

$$\vec{GM} = \frac{A_{\Delta BCM} - A_{\Delta ABM}}{A_{\Delta ABC}}\vec{GA} + \frac{A_{\Delta ACM} - A_{\Delta ABM}}{A_{\Delta ABC}}\vec{GB}$$

$$\vec{GI} = \frac{a-c}{a+b+c}\vec{GA} + \frac{b-c}{a+b+c}\vec{GB}$$ (3p)

M,G,I collinear $\Leftrightarrow \vec{GM}$ si $\vec{GI}$ $coliniari \Leftrightarrow \frac{A_{\Delta BCM} - A_{\Delta ABM}}{a-c} = \frac{A_{\Delta ACM} - A_{\Delta ABM}}{b-c}$

Hence the result appears (2p)

Problem 3

Be $n \in N^*$ determined.

a)Determine the rest of the division of 5^{2^n} to 2^{n+2}.

b)Show that the array (order) $a_n = \left(\left\{\frac{5^{2^n}}{2^{n+2}}\right\}\right)^{-1}$, where $\{a\}$ is the fractionary part of the real number a, is a geometric progression and that every other arithmetic progression $(b_n)_{n\geq 1}$ with natural terms that has two common terms with $(a_n)_{n\geq 1}$ has an infinity of common terms with the progression $(a_n)_{n\geq 1}$.

Solutions problem 3

a). $5^{2^n} = 5^{2^n} - 1 + 1 = (5-1)(5^{2^0}+1)(5^{2^1}+1)......(5^{2^{n1}}+1)+1$ 5p

$5-1 \vdots 4$ si $5^{2^k}+1 \vdots 2$ Hence the result appears 5p

b). $\left\{\frac{5^{2^n}}{2^{n+2}}\right\} = \left\{\frac{2^{n+2}k+1}{2^{n+2}}\right\} = \frac{1}{2^{n+2}} \Rightarrow \left\{\frac{5^{2^n}}{2^{n+2}}\right\}^{-1} = 2^{n+2}$ so the geometric progression 3p

$b_s = a_p \Rightarrow b_s = 2^{n+p}(the\_first\_two\_common\_terms)$

$b_s + mr = 2^{n+t}(the\_other\_two\_common\_terms)$ 2p

$mr = 2^{n+p}(2^{t-p}-1) \Rightarrow r|2^{n+p}(2^{t-p}-1) \Rightarrow r|2^{n+p}(2^{k(t-p)}-1)$ 5p

$2^{n+p+kt-kp} - 2^{n+p} = ru \Rightarrow 2^{n+p} + ru = 2^{n+p+k(t-p)} \Rightarrow$

$b_s + ru = a_p 2^{k(t-p)} \Rightarrow other\_common\_terms$ 5p

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Problem 4

Consider the array $(x_n)_{n\geq 0}$ given by $x_n = 2^n - n$, $n \in N$. Determine all the natural numbers p for which $s_p = x_0 + x_1 + x_2 + \cdot\cdot\cdot + x_p$ is a degree with a natural exponent of 2.

Solutions problem 4

$S_p=2^{p+1}-\frac{p(p+1)}{2}-1$ 5p

It is shown that $2^p<s_p<2^{p+1}$, for $p\geq 3$. s_p is a no a degree with a natural exponent of 2. 3p

It is proven through mathematical induction that $\frac{p(p+1)}{2}+1<2^p, for\ p\geq 3$ 8p

It is proven that $2^p<s_p$ 4p

It is noticed that $s_0=2^0$, $s_1=2^1$,$s_2=2^2$ 3p

Hence, p€{0,1,2} 2p