# MULTIDISCIPLINARY PROSOFT@NT CONTEST

March 2018

## MARKING SCHEME

## XI TH GRADE MATHEMATICS

### PROBLEM I

Let $k \geq 2$ a given natural even number and the function $f:\mathbb{R}\to\mathbb{R}$,

$$f(x)=\frac{C_k^1 x + C_k^3 x^3 + \ldots C_k^{k-1}x^{k-1}}{C_k^0 + C_k^2 x^2 + \ldots + C_k^k x^k}.$$

Calculate $\lim\limits_{n\to\infty}\Big(\underbrace{f\circ f\circ\ldots\circ f}_{n \text{ ori}}\Big)(x)$, where $x\in\mathbb{R}$.

**SOLUTION:**

It can be seen that $f(x)=\dfrac{(1+x)^k-(1-x)^k}{(1+x)^k+(1-x)^k}$, and by solving you get

$$(f\circ f)(x)=\frac{(1+x)^{k^2}-(1-x)^{k^2}}{(1+x)^{k^2}+(1-x)^{k^2}}.$$

……………..5 points.

Demonstrate by mathematical induction that $\Big(\underbrace{f\circ f\circ\ldots\circ f}_{n \text{ ori}}\Big)(x)=\dfrac{(1+x)^{k^n}-(1-x)^{k^n}}{(1+x)^{k^n}+(1-x)^{k^n}},\ \forall x\in R.$

……………...5 points.

- For $x>0$, we have $\left|\dfrac{1-x}{1+x}\right|<1$ and such $\lim\limits_{n\to\infty}\left[\left(\dfrac{1-x}{1+x}\right)^k\right]^n=0$ and, it results that,

$$\lim_{n\to\infty}\Big(\underbrace{f\circ f\circ\ldots\circ f}_{n \text{ ori}}\Big)(x)=\lim_{n\to\infty}\frac{1-\left(\dfrac{1-x}{1+x}\right)^{k^n}}{1+\left(\dfrac{1-x}{1+x}\right)^{k^n}}=1.$$

…………..5 points.

- For $x=-1$, $\lim\limits_{n\to\infty}\Big(\underbrace{f\circ f\circ\ldots\circ f}_{n\text{ ori}}\Big)(x)=\lim\limits_{n\to\infty}\dfrac{-2^{k^n}}{2^{k^n}}=-1.$ ……………...2 points.

- For $x<0,\ x\neq -1$, we have $\left|\dfrac{1+x}{1-x}\right|<1$ and $\lim\limits_{n\to\infty}\Big(\underbrace{f\circ f\circ\ldots\circ f}_{n\text{ ori}}\Big)(x)=\lim\limits_{n\to\infty}\dfrac{\left(\dfrac{1+x}{1-x}\right)^{k^n}-1}{\left(\dfrac{1+x}{1-x}\right)^{k^n}+1}=-1.$

………………..5 points.

- For $x=0$, $f(0)=0$ and $\lim\limits_{n\to\infty}\Big(\underbrace{f\circ f\circ\ldots\circ f}_{n\text{ ori}}\Big)(0)=0.$

Thus, $\lim\limits_{n\to\infty}\Big(\underbrace{f\circ f\circ\ldots\circ f}_{n\text{ ori}}\Big)(x)=\operatorname{sgn}(x)$, for any $x\in R$. ……………3 points.

(***)

**PROBLEM II**

Let matrices $A,B,C\in\mathbf{M}_n(\mathbb{C})$, such that $ABC=I_n$. Prove that, if matrices

$I_n+A+AB, I_n+B+BC, I_n+C+CA$ are invertible, then the sum of their inverses equals $I_n$.

**SOLUTION:**

$(I_n+A+AB)^{-1}+(I_n+B+BC)^{-1}+(I_n+C+CA)^{-1}=$

$(I_n+A+AB)^{-1}+(I_n+B+BC)^{-1}\cdot A^{-1}\cdot A+(I_n+C+CA)^{-1}\cdot(AB)^{-1}\cdot AB=$ …………7 points.

$(I_n+A+AB)^{-1}+(A+AB+ABC)^{-1}\cdot A+(AB(I_n+C+CA))^{-1}\cdot AB=$ …………7 points.

$\left(I_n + A + AB\right)^{-1} + \left(A + AB + I_n\right)^{-1} \times A + \left(AB + I_n + A\right)^{-1} \times AB =$ …………7 points.

$\left(I_n + A + AB\right)^{-1}\left(I_n + A + AB\right) = I_n$ …………4 point.

(***)

**PROBLEM III**

Calculate: $\lim\limits_{n\to\infty} \dfrac{1}{(n!)^{\frac{1}{n}}} \cdot \log_2\left(\sqrt[2^n]{2} + \sqrt[2^n]{3} + ....... + \sqrt[2^n]{2^n+1}\right)$.

Solution

Using Bernoulli inequality we have $1 < \sqrt[2^n]{1+k} < 1 + \dfrac{k}{2^n} \Rightarrow 2^n < \sum\limits_{k=1}^{2^n} \sqrt[2^n]{1+k} < \dfrac{3\cdot 2^n + 1}{2} < 2^{n+1}$ …10 points

$\Rightarrow n < \log_2\left(\sum\limits_{k=1}^{2^n} \sqrt[2^n]{1+k}\right) < n+1 \Rightarrow \dfrac{n}{\sqrt[n]{n!}} < \dfrac{1}{\sqrt[n]{n!}} \cdot \log_2\left(\sum\limits_{k=1}^{n} \sqrt[2^n]{1+k}\right) < \dfrac{n+1}{\sqrt[n]{n!}}$. ……5 points

But because $\lim\limits_{n\to\infty} \dfrac{n}{\sqrt[n]{n!}} = \lim\limits_{n\to\infty} \dfrac{n+1}{\sqrt[n]{n!}} = e$, it results that the limit of the series is e. ….10 points

**PROBLEM IV**

Determine the functions $f: [0,3] \to (0,1]$ which respect/follow the conditions:

i. $\lim\limits_{x\to 0} \dfrac{f(x) - f(0)}{x^2} \in ¡$

ii. $f(3x) + 3f(x) = 4f^3(x),\ \forall x \in [0,1]$

**SOLUTION**

For $x = 0 \Rightarrow f(0) + 3f(0) = 4f^3(0) \Rightarrow f(0) \in \{-1,0,1\} \Rightarrow f(0) = 1 \in (0,1]$

We have $\lim_{x\to 0}(f(x) - f(0)) = \lim_{x\to 0}\frac{f(x)-f(0)}{x^2} \cdot x^2 = 0 \Rightarrow f$ continuous in $x_0 = 0$ **…..2 points**

Let $g: [0,3] \to \left[0,\frac{\pi}{2}\right), g(x) = \arccos f(x), (\forall) x \in [0,3] \Rightarrow g$ continuous in $x_0 = 0$ and

$f(x) = \cos g(x), (\forall)x \in [0,3].$ ….2 points

Replacing in *ii)*, we get:

$\cos g(3x) = 4\cos^3 g(x) - 3\cos g(x) = \cos 3g(x)$ …..2 points

Because $g(x) \in \left[0,\frac{\pi}{2}\right) \Rightarrow \cos g(3x) \geq 0$, it also results that $\cos 3g(x) \geq 0$ (1) …..2 points

Because $g(x) \in \left[0,\frac{\pi}{2}\right)$, $3g(x) \in \left[0,\frac{3\pi}{2}\right)$ (2)

From (1) and (2) we get $3g(x) \in \left[0,\frac{\pi}{2}\right)$, so the equation $\cos g(3x) = \cos 3g(x)$ implies

$g(3x) = 3g(x), (\forall)x \in [0,1] \Rightarrow g(x) = 3g\left(\frac{x}{3}\right), (\forall)x \in [0,3]$

By mathematical induction it results that $g(x) = 3^n g\left(\frac{x}{3^n}\right), (\forall)x \in [0,3]$ and $(\forall)n \in \mathbb{N}^*$, it results that

$\frac{g(x)}{x} = \frac{g\left(\frac{x}{3^n}\right)}{\frac{x}{3^n}}$ $(\forall)x \in (0,3]$ şi $(\forall)n \in \mathbb{N}^*$ (3) …. 6 points

Because $g(0) = \arccos f(0) = \arccos 1 = 0$, we have:

$$\lim_{x\to 0}\frac{f(x) - f(0)}{x^2} = \lim_{x\to 0}\frac{\cos g(x) - 1}{x^2} = \lim_{x\to 0}\frac{-2\sin^2\frac{g(x)}{2}}{\left(\frac{g(x)}{2}\right)^2} \cdot \frac{\left(\frac{g(x)}{2}\right)^2}{x^2} = -\frac{1}{2}\lim_{x\to 0}\left(\frac{g(x)}{x}\right)^2 \in \mathbb{R}$$

…. 6 points

Let $\lim\limits_{x\to 0}\left(\frac{g(x)}{x}\right)^2 = a \in [0,\infty)$. We get:

$\lim_{n\to\infty}\left(\frac{g(x)}{x}\right)^2 = \lim\limits_{n\to\infty}\left(\frac{g\left(\frac{x}{3^n}\right)}{\frac{x}{3^n}}\right)^2 = a, (\forall)x \in (0,3]$ și $g(x) \geq 0 \Rightarrow \frac{g(x)}{x} = \sqrt{a} \Rightarrow g(x) = \sqrt{a}x,$

$(\forall)x \in (0,3]$

Because $0 \leq g(x) < \frac{\pi}{2}, (\forall)x \in [0,3] \Rightarrow 0 \leq \sqrt{a}x < \frac{\pi}{2}, (\forall)x \in [0,3] \Rightarrow 0 \leq \sqrt{a} < \frac{\pi}{6}$ …3 points

and such, the functions that respect the hypothesis are $f: [0,3] \to (0,1], f(x) = \cos mx,$ where $m \in [0, \frac{\pi}{6})$. …. 2points