Max Minus Min (maxminmin)

Author: Vlad-Mihai Bogdan

Developer: Vlad-Mihai Bogdan

Solution

One can see that the formula

$$f(K) = \sum_{S \subseteq \{1,2,\ldots,N\}, |S|=K} (\max(S) - \min(S))$$

can be rewritten as

$$f(K) = \sum_{S \subseteq \{1,2,\ldots,N\}, |S|=K} \max(S) - \sum_{S \subseteq \{1,2,\ldots,N\}, |S|=K} \min(S)$$

Since we are working with all number from 1 to N, the sets of minimum values is in a way symmetric with the set of maximum values. More exactly, i will be the minimum element in subsets of size K as often as $N - i + 1$ will be the maximum element in such subsets. Therefore, it's enough to calculate the sum of minimums for each K. Let this value be $s(K)$. The sum of maximum values will be $\binom{N}{K} \cdot (N+1) - s$.

Now, for an $O(N^2)$ solution, for each value of K, we will fix the minimum value. Let this value be i. In that case

$$s(K) = \sum_{i \in \{1,2,\ldots,N\}} \binom{N-i}{K-1} \cdot i$$

In order to get full marks, we need to calculate the above sum faster. If we rewrite the terms and then apply the Hockey-stick identity, we will see the above sum is equal to $\binom{N+1}{K+1}$. This way, we can calculate $s(i)$ in $O(1)$ for each value of K, giving us an $O(N)$ solution.

One can also find a bijection between the two sets described by the sum above and the set of $K+1$ items out of $N+1$. The proof for this solution is left as an exercise for the reader.